Indian Statistical Institute
Bangalore Centre
B.Math (Hons.) III Year 2010-2011
First Semester
Statistics III

Mid-semester Examination Date : 1.10.10

Answer as many questions as possible. The maximum you can score is 60
State clearly the results you use.

1. Consider the linear model

$$Y(n \times 1) = X(n \times p)\, \beta(p \times 1) + \varepsilon(n \times 1),$$

where $E(\varepsilon) = 0$ and $Cov(\varepsilon) = \sigma^2 I_n$.

(a) Suppose l is in R^p. When is $l'\beta$ said to be estimable? Prove that $l'\beta$ is estimable if and only if

$$l' = l'S^{-}S, \ S = X'X.$$

(b) What do you mean by a "least square estimate" of a parametric function ? Does it always exist ? Is it unique ?

(c) Suppose $\hat{\beta}$ denote a least square estimate of β. If $l'\beta$ is estimable prove that $l'\hat{\beta}$ has minimum variance among all linear unbiased estimators of $l'\beta$.

(d) While working with a linear model with three parameters $\beta_0, \beta_1, \beta_2$, one came came across the system of normal equations $S\beta = Z$, where $Z = (5, -6, -3)'$ and

$$S = \begin{bmatrix} 10 & -5 & -5 \\ -5 & 3 & 2 \\ -5 & 2 & 3 \end{bmatrix}.$$

Find two g-inverses G and H of S. Suppose $\hat{\beta} = GZ$ and $\tilde{\beta} = HZ$. Compute $\hat{\beta}_1 - \hat{\beta}_2$, $\tilde{\beta}_1 - \tilde{\beta}_2$, $\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2$ and $\tilde{\beta}_0 + \tilde{\beta}_1 + \tilde{\beta}_2$. Explain the fact that the first two numbers are same while the last two are not.

[(2+3) + (2+2+2) + 4 + (2 x 2 + 4 + 4) = 27]

2. Consider the linear model of Q1 and assume that ε follows multivariate normal distribution. Consider a vector $H'\beta$ of k linearly independent estimable linear functions of β.

(a) Assuming $\hat{\beta}$ to be the same as in Q1(c), obtain $Cov(H'\hat{\beta})$.

(b) Show that $Cov(H'\hat{\beta})$ is nonsingular. [3 + 5 = 8]

3. Let $X_1, X_2, \cdots X_n$ be i.i.d. $N(0,1)$ variables. Let $X = (X_1, X_2, \cdots X_n)'$.

(a) Suppose $Q = X'AX, Q_1 = X'BX$ and $Q_2 = Q - Q_1$ is non-negative for every value of X. If $Q \sim \chi^2(a)$ and $Q_1 \sim \chi^2(b)$ then, show that $Q_2 \sim \chi^2(a-b)$.

(b)Suppose $Q_i = X'A_iX,\ i = 1,2$ and both Q_1 and Q_2 follow χ^2 distribution. Prove that a necessary and sufficient condition that Q_1 and Q_2 are independent is that $A_1A_2 = 0$.

[6 + 5 =11]

4. (a) When is a vector of random variables said to follow multivariate normal distribution ? Derive the density function of a p-variate normal variable having a nonsingular covariance matrix.

[2 + 4 =6]

(b) Suppose $X_1, X_2, \cdots X_n$ are i.i.d random variables following $N_p(\mu, \Sigma)$ distribution. Let $\bar{X} = \sum_{i=1}^n X_i$ and $S = \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})'$.

(i) Show that $\bar{X}$ and S are independently distributed.

(ii) Let $V = S/(n-1)$ and $T_0^2 = n(\bar{X} - \mu_0)'V^{-1}(\bar{X} - \mu_0)$. Show that there is a constant c involving n and p such that cT_0^2 follows F distribution, central if $\mu = \mu_0$.

(iii) We want to test the hypothesis $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Let $L(\mu, \Sigma)$ denote the likelihood function and $\hat{\Sigma_0} = (1/n)\sum_{i=1}^n (X_i - \mu_0)(X_i - \mu_0)'$. Assuming that

$sup_{\mu=\mu_0} L(\mu, \Sigma) = L(\mu_0, \hat{\Sigma_0})$,

show how you can use likelihood ratio test procedure for testing H_0 against H_1. [Hint : The result in (ii) may be useful.]

[6 + 4 + 10 = 20]